Indian Statistical Institute
B. Math. Hons. I Year
Semestral Examination 2002-2003
Algebra II

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You may **use** and **quote** any result proved in **class room** and in **assignments with out proof.**
Total marks: 55
Section I: Answer any four and each question carries five marks

(1) Let G be a finite group. A subgroup H of G is called **characteristic in** G if H is fixed by all automorphisms of G, that is, $T(H) = T$ for all automorphisms T. For a Sylow p-subgroup H of G,

(i) H is normal in G if and only if H is characteristic in G and

(ii) $N(H) = \{x \in G \mid xHx^{-1} = H\}$ is normal in G if and only if H is normal in G.

(2) (i) Let V be a finite-dimensional hermitian space. Let $T: V \to V$ be an idempotent linear operator, that is, $T^2 = T$. Show that T is self-adjoint if and only if T is normal, that is $TT^* = T^*T$.

(ii) If V has dimension 2 with standard basis e_1 and e_2 and T is a linear operator such that $T(e_1) = (1+i, 2)$ and $T(e_2) = (i, i)$. Find the matrices (with respect to the standard basis) of T.

(3) Prove that any finite integral domain has only p^k elements for some prime integer p and an integer $k \geq 1$.

(4) Let R be a pid and $a, b \in R$. Suppose a and b are irreducible. Then either $(a) + (b) = R$ or a and b are associates.

(5) Let R be a ring. An R-module V is called *indecomposable* if V can not be written as a sum two non-zero submodules of V. Prove that

(i) any simple R-module is indecomposable and

(ii) for a prime integer p, the $\mathbb{Z}$-module, $\mathbb{Z}/(p^n)$ is indecomposable and for $n > 1$, $\mathbb{Z}/(p^n)$ is not simple.

Section II : Answer all questions

(1) (a) Let R be a commutative ring in which every subring is an ideal. Then $R \simeq \mathbb{Z}$ or $\mathbb{Z}/(n)$ and compute the nilradical of R in each case.

(b) Let R be a pid and $a, b \in R$. Assume that not both a, b are zero. If (a, b) and $[a, b]$ denote the gcd and lcm of a and b, then $(a, b)[a, b]$ is an associate of ab.

Marks: $8 + 5 = 13$

(2) (a) Determine the presentation matrix for the ideal $(2, 1 + \sqrt{(-5)})$ in $\mathbb{Z}[\sqrt{(-5)}]$.

(b) Classify upto similarity all $n \times n$ matrices A such that $A^n = I$.

(c) Find an isomorphic product of cyclic groups when V is the abelian group generated by x, y, z with relations

$$7x + 5y + 2z = 0 \quad 3x + 3y = 0 \quad 13x + 11y + 2z = 0.$$

Marks: $5 + 12 + 5 = 22$